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A Covariant Method for Calculating Helicity Amplitudes

Roberto Vega [1]
Department of Physics
Southern Methodist University
Dallas, TX 75275

Jose Wudka [2]
Department of Physics
University of California at Riverside
Riverside, CA 92521-0413

Abstract

We present an alternative approach for calculating helicity amplitudes for processes involving both massless and massive fermions. With this method one can easily obtain covariant expressions for the helicity amplitudes. The final expressions involve only four vector products and are independent of the basis for gamma matrices or specific form of the spinors. We use the method to obtain the helicity amplitudes for several processes involving top quark production.

[1] vega@mail.physics.smu.edu
[2] jose.wudka@ucr.edu

1 Introduction

The conventional approach used to determine decay rates and cross sections is to square the Feynman amplitude, average over initial, and sum over final spin states. This approach is very inconvenient to implement if the number of Feynman diagrams and the number of final state particles is large. This is true even with the use of all the modern tools for symbolic computation. The conventional approach also has the drawback that in summing over spin states one loses information on spin correlations. Such correlations may be important for exploring new physics at future colliders.

An alternative approach, which has gained popularity in the past decade, is to compute the helicity amplitudes numerically. By doing this, spin correlations are easily explored and the process of squaring the amplitude is trivial. The idea of calculating helicity amplitudes is probably as old as the conventional approach and might even pre-date it[1, 2]. More recently, motivated by the need to consider more complicated processes, a number of novel methods[3, 4] for obtaining helicity amplitudes have been developed. Most of these, however, are viable only in the limit where fermion masses can be ignored. In this limit helicity and chirality are equivalent, leading to enormous simplifications; this is particularly evident in processes involving the W-gauge bosons. Most of the recent so-called helicity methods[3, 4, 5] take advantage of these simplifications[6].

Although, there have been several attempts to generalize the helicity methods for massless fermions to the massive case, the resulting procedure is rather cumbersome and requires the introduction of arbitrary reference momenta[7]. It is the purpose of this paper to present a simple and systematic method for calculating helicity amplitudes which works for processes involving both massless and massive fermions. The approach that we present leads to a covariant expression for the helicity amplitude in terms of a minimum number of four vector products and is independent of the basis for Dirac matrices or specific forms of the spinors. Thus, we believe that our approach leads to an analytic expression which is relatively simple and easy to compare with.

For illustration we apply our method to compute the helicity amplitudes for

various processes involving the top quark. We consider the processes:

$$e^+e^- \to t\bar{t}$$
$$\gamma\gamma \to t\bar{t}$$

2 The Method

When evaluating a Feynman amplitude involving a pair fermions and several gauge bosons, in the initial or final state, the amplitude is expressed as a sum of terms which have the form,

$$\bar{u}(p,\lambda)\not{B}_0\not{B}_1\not{B}_2\cdots\not{B}_{2n+1}\,u(p',\lambda'), \tag{1}$$

where (p,λ) and (p',λ') are the momenta and helicity of the external fermions. The particular labeling of the indices is chosen for convenience in the manipulations that follow. Note that, as written, (1) has an even number of γ-matrices sandwiched between spinors. The case where the number of γ-matrices is odd is treated by setting $\not{B}_0 = I$.

In the usual approach one squares these terms and sums over the fermion spins to obtain a sum over terms of the form,

$$tr[(\not{p}+m)\not{B}_{2n+1}\not{B}_{2n}\cdots\not{B}_1\not{B}_0\,(\not{p}'+m')\not{B}_0\not{B}_1\not{B}_2\cdots\not{B}_{2n+1}]. \tag{2}$$

The traces can then be evaluated and the result will be a function of Minkowski products of the external particle four momenta. The expression is then easily integrated over phase space by use of Monte Carlo methods. In this approach one does not need explicit values of components of free spinors (which depend on the particular representation used for the γ-matrices). This method is also convenient because, in addition to getting rid of the γ-matrices and expressing the amplitude in terms of simple "dot" products, the final result is independent of the unobservable fermion spins.

In complicated cases where the amplitude involves the sum over a large number of Feynman diagrams, and each diagram involves many fermion-gauge boson vertices, the number of traces and the length of the traces may render the standard approach impractical. The standard approach is also inconvenient in cases where interference effects are of interest and one may want

to explicitly display the relative phases between the different Feynman diagrams. In these cases it is more convenient to obtain an expression for the amplitude corresponding to each graph which is simple to evaluate numerically. The square of the amplitude is then obtained by just squaring a single complex number.

For processes that only involve massless or nearly massless fermions there have been many papers where the so called helicity methods for evaluating tree level amplitudes are developed[3, 4, 5]. In contrast to the standard approach, the amplitude in the Helicity Method is expressed in terms of spinor products of the form

$$s(p,p') = \bar{u}(p,+)u(p',-).$$

In this definition the p and p' are light-like four-vectors. The $\pm$ refer to the helicity of the fermion. Note that $|s(p,p')|^{2} = p{\cdot}p'$, so that up to a phase $s(p,p')$ contains essentially the same information as $p{\cdot}p'$. These methods prove relatively simple to implement and lead to compact expressions valid whenever the fermion masses can be ignored. Generalizations of this approach to the massive fermion case[7, 8] exist, but the results are cumbersome: the method requires the introduction of extra light-like momenta and the expression of the spinor for a massive particle as a sum over spinors for massless particles. Consequently, as the number of external particles increases, the number of terms for each diagram grows and results quickly become unmanageable and more prone to human error. In another approach[9] some authors found it more convenient to obtain the amplitudes by performing all the γ-matrix products numerically. Although this approach appears to be efficient, the result is not expressible in a simple compact analytic form which one can study. We finally note that all the approaches quoted above require writing down particular expressions for the spinors.

We will see below that with the approach presented here, it is possible to actually write the amplitude in terms of "dot" products without the use of particular expressions for the spinors. The procedure is relatively simple and leads to compact analytic expressions which are easy to compare with and put into code to evaluate numerically or symbolically.

The basic idea is to observe that (1) can be re-written as the trace,

$$tr[u(p',\lambda') \otimes \bar{u}(p,\lambda) \not{B}_0 \not{B}_1 \not{B}_2 \cdots \not{B}_{2n+1}]. \tag{3}$$

Although this trivial observation has been used by several authors in the past[10, 11]. there was no attempt was made to make use of the Michael-Bouchiat identity (as we do below) to obtain an expression for $u(p',\lambda')\bar{u}(p,\lambda)$. The object, $u(p',\lambda')\otimes\bar{u}(p,\lambda)$, is a four by four matrix in Dirac space and can be expressed in terms of an orthogonal basis of the four-dimensional Dirac space [12]. We do not follow this approach here because we found it simpler to make use of a spinor identity due to Michel and Bouchiat [13],

$$u(p,\lambda')\bar{u}(p,\lambda) = \frac{1}{2}(\not{p}+m)(\delta_{\lambda\lambda'} + \gamma_5 \not{\eta}^i \sigma^i_{\lambda\lambda'}). \tag{4}$$

The σ's are the three Pauli matrices and there is an implied sum over $i = 1, 2, 3$. Note that in the case where $\lambda' = \lambda$ expression (4) reduces to the usual projection operator for a state of momentum p and helicity λ. The η's are defined such that,

$$\begin{aligned} \eta_i{\cdot}\eta_j &= -\delta_{ij} \\ \eta_i{\cdot}p &= 0. \end{aligned} \tag{5}$$

They are the spin vectors corresponding to the four momentum p. For,

$$\vec{p} = |\vec{p}|\ (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

a standard representation of the η's is given by,

$$\begin{aligned} \eta_1^\mu &= (0; \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \eta_2^\mu &= (0; -\sin\phi, \cos\phi, 0) \\ \eta_3^\mu &= (\frac{|\vec{p}|}{m}; \frac{E}{m}\frac{\vec{p}}{|\vec{p}|}), \end{aligned} \tag{6}$$

where $\eta_k^\mu = (0; \delta_k^i)$ for the case when $|\vec{p}| = 0$.

For completeness we have included a proof of identity (4) in the appendix. The identity is only valid if both spinors correspond to the same momenta. To make use of it we must generalize it to obtain an expression for $u(p',\lambda')\bar{u}(p,\lambda)$. If $m' = m$ this could be done by applying a boost operator to both sides of (4). It is, however, simpler and more general to observe that any spinor $u(p',\lambda')$ can be expressed in the form

$$u(p',\lambda') = \mathcal{N}\sum_\lambda C_{\lambda'\lambda}(\not{p}'+m')u(p,\lambda), \tag{7}$$

where $p'^2 = m'^2$, and $\mathcal{N}$ is a normalization factor. This expression is valid whether or not $m' = m$. Note that the helicity index λ is defined with respect to the direction of $\vec{p}$, while λ' is defined with respect to $\vec{p}\,'$; i.e. the frames in which λ and λ' are defined are rotated with respect to each other. Multiplying (7) on the right by $\bar{u}(p,\lambda)$ and using identity (4) we obtain,

$$C_{\lambda'\lambda} = \mathcal{N}\,\bar{u}(p,\lambda)u(p',\lambda'), \tag{8}$$

with,

$$\mathcal{N} = \frac{1}{\sqrt{2(p{\cdot}p' + mm')}}.$$

It is easy to verify that the matrix $C_{\lambda'\lambda}$ is unitary in helicity space as it should be. Using (7) in (4) we obtain the desired generalization,

$$u(p',\lambda') \otimes \bar{u}(p,\lambda) = \frac{\mathcal{N}}{2} \sum_{\lambda''} C_{\lambda'\lambda''}\,(\not{p}' + m')(\not{p} + m)(\delta_{\lambda\lambda''} + \gamma_5 \not{\eta}^i \sigma^i_{\lambda\lambda''}). \tag{9}$$

Since $C_{\lambda\lambda'}$ is unitary it drops out of our expressions upon summing the square of the helicity amplitudes over all helicities. Thus, if in the end one is going to sum over all helicities one can simplify (9) by setting $C_{\lambda'\lambda''} = \delta_{\lambda'\lambda''}$. The resulting amplitudes will not correspond to the correct helicity amplitudes but the process of squaring them and summing over all helicity indices will indeed generate the correct result for the sum of the squares.

If spin correlations are of interest then one needs to make use of the specific expressions for the $C_{\lambda\lambda'}$. The expressions follow from (8)

$$\begin{aligned}
C_{++} &= \left[\frac{EE' - \,|\vec{p}|\,\ |\vec{p}\,'|\, + mm'}{p{\cdot}p' + mm'}\right]^{1/2} [\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}\ \mathrm{e}^{i(\phi'-\phi)}\ ] \\
C_{-+} &= \left[\frac{EE' + \,|\vec{p}|\,\ |\vec{p}\,'|\, + mm'}{p{\cdot}p' + mm'}\right]^{1/2} [\cos\frac{\theta}{2}\sin\frac{\theta'}{2}\ \mathrm{e}^{-i\phi'}\ - \sin\frac{\theta}{2}\cos\frac{\theta'}{2}\ \mathrm{e}^{-i\phi}\ ] \\
C_{--} &= C^*_{++} \\
C_{-+} &= -C^*_{+-}.
\end{aligned} \tag{10}$$

The angles (θ,ϕ) and (θ',ϕ') correspond to the axial and azimuthal angles for $\vec{p}$ and $\vec{p}\,'$ respectively. In the center of momentum frame where $\vec{p}\,' = -\vec{p}$

the expressions for $C_{\lambda\lambda'}$ simplify to,

$$\begin{aligned} C_{\pm\pm} &= 0 \\ C_{\pm\mp} &= \mp\, e^{\pm i\phi}\ . \end{aligned} \tag{11}$$

This remains true whether or not $m = m'$. If in addition it is also true that $m = m'$ then (9) simplifies to

$$\begin{aligned} u(p', \lambda) \otimes \bar{u}(p, \lambda) &= -\lambda \frac{1}{2} \gamma_0 (\not{p} + m) \gamma_5 \not{\eta}^\lambda \ e^{i\lambda\phi} \\ u(p', \lambda) \otimes \bar{u}(p, -\lambda) &= -\lambda \frac{1}{2} \gamma_0 (\not{p} + m)(1 - \lambda\gamma_5 \not{\eta}^3)\ e^{i\lambda\phi}\ , \end{aligned} \tag{12}$$

where

$$\eta^\lambda = \eta_1 - i\lambda\, \eta_2. \tag{13}$$

These results also follow by using the relation[6]

$$u(p, \lambda) = -\lambda\gamma_0\, u(-p, -\lambda),$$

in expression (4).

Finally we remark that in the limit $m \to 0$ equations (9) and (12) still apply with the proviso that $\eta_3^\mu \approx p^\mu/m + \mathcal{O}(m/E)$. In the massless limit, where $m', m \to 0$, expression (9) simplifies to

$$u(p', \lambda')\bar{u}(p, \lambda) = \begin{cases} \dfrac{1}{2\sqrt{2p{\cdot}p'}}\, C_{\lambda'\lambda}\, \not{p}'\not{p}\,(1 + \lambda\gamma_5) & \text{for } \lambda' = -\lambda, \\ \dfrac{1}{2\sqrt{2p{\cdot}p'}}\, C_{\lambda(-\lambda)}\, \not{p}'\not{p}\, \gamma_5 \not{\eta}^\lambda & \text{for } \lambda' = \lambda. \end{cases} \tag{14}$$

It is easy to verify that (14) satisfies all of the identities valid for the inner product of two massless spinors. For example,

$$\bar{u}(p, -)\, u(p', +) = -(\bar{u}(p, +)\, u(p', -))^*,$$

and,

$$|\bar{u}(p, +)\, u(p', -)|^2 = 2p \cdot p'.$$

Equation (9), and its simplified forms (12) and (14), play a central role in our method. With them we can evaluate the trace in (3) and obtain an analytic expression for the amplitude in terms of dot products.

Evaluating the trace, even at the amplitude level, may lead to very complicated expressions if the number of external gauge bosons connected to a single fermion line is large. Therefore, we develop an iterative method for obtaining this trace. The iterations can be easily performed either symbolically (using code like REDUCE or FORM) or numerically. The basic idea is to use the fact that any product of an odd number of gamma matrices can be written in the form,

$$\not{B}_1 \not{B}_2 \cdots \not{B}_{2n+1} = \not{V}_n + \gamma_5 \not{A}_n. \tag{15}$$

Where the V's and A's are obtained iteratively from,

$$\begin{aligned} V_n^\mu &= \frac{1}{4} tr[\gamma^\mu(\not{V}_{n-1} + \gamma_5 \not{A}_{n-1}) \not{B}_{2n} \not{B}_{2n+1}] \\ A_n^\mu &= \frac{1}{4} tr[\gamma^\mu(\gamma_5 \not{V}_{n-1} + \not{A}_{n-1}) \not{B}_{2n} \not{B}_{2n+1}], \\ V_\circ^\mu &= B_1^\mu \\ A_\circ^\mu &= 0. \end{aligned} \tag{16}$$

This result follows after repeated use of the γ identity[3],

$$\gamma_\mu \gamma_\nu \gamma_\sigma = g_{\mu\nu}\gamma_\sigma + g_{\nu\sigma}\gamma_\mu - g_{\mu\sigma}\gamma_\nu + i\gamma_5 \epsilon_{\mu\nu\sigma\rho}\gamma^\rho.$$

For convenience we define the following expressions,

$$\begin{aligned} F_{\lambda,\lambda'}(V_n, A_n) &= \bar{u}(p,\lambda)(\not{V}_n + \gamma_5 \not{A}_n) u(p',\lambda') \\ F_{\lambda,\lambda'}(B_0, V_n, A_n) &= \bar{u}(p,\lambda) \not{B}_0 (\not{V}_n + \gamma_5 \not{A}_n) u(p',\lambda'). \end{aligned} \tag{17}$$

The essence of the method is to write expressions of the form (1) in terms of $F_{\lambda,\lambda'}(V_n, A_n)$ and $F_{\lambda,\lambda'}(B_0, V_n, A_n)$ defined above. The V_n's and A_n's are obtained by the simple iteration process (16). Finally, using (9) we express (17) in terms of four vector products. Since the general expressions for

[3] In our conventions $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon_{0123} = +1$.

$F_{\lambda,\lambda'}(V_n, A_n)$ and $F_{\lambda,\lambda'}(B_0, V_n, A_n)$ are common to every Feynman diagram we display their form below.

$$\begin{aligned} F_{\lambda,\lambda'}(V,A) &= 2\mathcal{N}\sum_{\lambda''} C_{\lambda',\lambda''} M_{\lambda'',\lambda} \\ F_{\lambda,\lambda'}(B_0,V,A) &= 2\mathcal{N}\sum_{\lambda''} C_{\lambda',\lambda''} N_{\lambda'',\lambda} \end{aligned} \tag{18}$$

where

$$M_{\lambda'',\lambda} = \begin{cases} m'p{\cdot}V + mp'{\cdot}V \\ \qquad -\lambda[mm'\eta^3{\cdot}A + \eta^3{\cdot}W(p,p',A,V)] & \text{for } \lambda'' = \lambda \\ -\Big[mm'e^\lambda{\cdot}A + e^\lambda{\cdot}W(p,p',A,V)\Big] & \text{for } \lambda'' = -\lambda, \end{cases} \tag{19}$$

$$N_{\lambda'',\lambda} = \begin{cases} B_0 \cdot W(p,p',V,A) + mm'B_0 \cdot V \\ \qquad +\lambda\, B_0 \cdot W(\eta^3, m'p + mp', A, V) & \text{for } \lambda = \lambda \\ B_0 \cdot W(\eta^\lambda, m'p + mp', A, V) & \text{for } \lambda'' = -\lambda, \end{cases} \tag{20}$$

and,

$$W^\mu(x,y,z,w) = x^\mu y{\cdot}z + z^\mu x{\cdot}y - y^\mu x{\cdot}z + i\epsilon^{\mu\nu\rho\sigma} x_\nu y_\rho w_\sigma \ . \tag{21}$$

Of course these expressions take on much simpler forms when (12) and (14) are applicable. We will see an example of this simplification in the next section. We now have at hand all the tools necessary to express any object of the form (3) in terms of four vector products. In the next section we illustrate how these techniques are applied to compute the cross section for various processes.

3 Applications

As an illustration of our method we present the helicity amplitudes for two processes involving the top quark, namely,

$$e^+e^- \to t\bar{t}$$

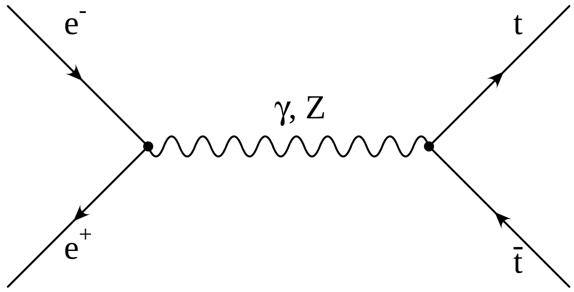

Figure 1: Feynman diagrams for the process $e^+e^- \rightarrow t\bar{t}$

and,

$$\gamma\gamma \rightarrow t\bar{t}.$$

These processes will be important in the study of the phenomenology of the top quark at future electron-positron colliders.

We can write the helicity amplitudes for both processes in terms of the spinor objects,

$$\begin{aligned} A_\mu(p,\lambda;p',\lambda') &\equiv \bar{u}(p,\lambda)\gamma_\mu\, v(p',\lambda') \\ B_\mu(p,\lambda;p',\lambda') &\equiv \bar{u}(p,\lambda)\gamma_\mu\gamma_5\, v(p',\lambda') \end{aligned} \tag{22}$$

We will work in the center of momentum system with the initial particles moving along the z-axis; we also take $m = m'$. In this frame we can then use (12) to obtain simple expressions for A_μ and B_μ,

$$\begin{aligned} A_\mu(p,\lambda;p',\lambda') &= \begin{cases} -2m\lambda\, g_{\mu i}\,\hat{p}^i & \text{if } \lambda' = \lambda \\ -2E\eta^\lambda_\mu(p) & \text{if } \lambda' = -\lambda, \end{cases} \\ B_\mu(p,\lambda;p',\lambda') &= \begin{cases} -2m\, g_{o\mu} & \text{if } \lambda' = \lambda, \\ -2\lambda\,|\vec{p}|\,\eta^\lambda_\mu & \text{if } \lambda' = -\lambda. \end{cases} \end{aligned} \tag{23}$$

Where $\eta^\lambda_\mu(p)$ are the polarization vectors defined in (13) corresponding the momentum p, mass m and helicity index λ.

The $e^- e^+ \to t\bar{t}$ Process:

We now have the necessary ingredients to obtain expressions for the helicity amplitudes. The Feynman diagrams for the first process are shown in figure 1. We take the $\gamma t\bar{t}$, $Zt\bar{t}$, and Ze^-e^+ couplings to be of the form[4],

$$\Gamma^{\mu}_{\gamma t\bar{t}} = q_t \gamma^{\mu}, \qquad \Gamma^{\mu}_{zt\bar{t}} = \gamma^{\mu}(c + d\gamma_5) \tag{24}$$

$$\Gamma^{\mu}_{\gamma e\bar{e}} = e\gamma^{\mu}, \qquad \Gamma^{\mu}_{ze\bar{e}} = \gamma^{\mu}(a + b\gamma_5). \tag{25}$$

The amplitudes can then be written in terms of (23),

$$\begin{aligned}
T_{\gamma}(\lambda_e, \lambda_{\bar{e}}; \lambda_t, \lambda_{\bar{t}}) &= -\frac{e\, q_t}{s} A_{\mu}(\lambda_e, \lambda_{\bar{e}})^{\dagger}\, A^{\mu}(\lambda_t, \lambda_{\bar{t}}) \\
T_z(\lambda_e, \lambda_{\bar{e}}; \lambda_t, \lambda_{\bar{t}}) &= -\frac{1}{s - m_z^2} \Big\{ ac \left[A_{\mu}(\lambda_e, \lambda_{\bar{e}})^{\dagger}\, A^{\mu}(\lambda_t, \lambda_{\bar{t}}) \right] \\
&\qquad + ad \left[A_{\mu}(\lambda_e, \lambda_{\bar{e}})^{\dagger}\, B^{\mu}(\lambda_t, \lambda_{\bar{t}}) \right] \\
&\qquad + bc \left[B_{\mu}(\lambda_e, \lambda_{\bar{e}})^{\dagger}\, A^{\mu}(\lambda_t, \lambda_{\bar{t}}) \right] \\
&\qquad + bd \left[B_{\mu}(\lambda_e, \lambda_{\bar{e}})^{\dagger}\, B^{\mu}(\lambda_t, \lambda_{\bar{t}}) \right] \Big\}.
\end{aligned} \tag{26}$$

For simplicity we have suppressed the momentum indices, but the momentum dependence should be clear from the context. The electron and positron momenta are designated by p_e and $p_{\bar{e}}$, and the top, anti-top quark momenta are designated by p and $\bar{p}$. In the CM frame the momenta take the form[5],

$$\begin{aligned}
p_e &= (E; 0, 0, E) \\
p_{\bar{e}} &= (E; 0, 0, -E) \\
p &= (E;\ |\vec{p}|\ \cos\theta, 0,\ |\vec{p}|\ \sin\theta) \\
\bar{p} &= (E; -\vec{p}),
\end{aligned} \tag{27}$$

[4] In the standard model the parameters q_t, a, b, c, and d are given by

$$a = e\,\frac{4s_w^2 - 1}{4s_w c_w}, \qquad c = e\,\frac{1 - 8s_w^2/3}{4s_w c_w}, \qquad b = -d = \frac{e}{4s_w c_w}, \qquad q_t = -\frac{2e}{3}.$$

[5] These are contravariant four vectors.

where E is the energy of the incoming electron in the center of mass frame. The electron mass is ignored in all of our expressions. Using these expressions for the momenta together with (23) and (26) it is simple to obtain an expression for the helicity amplitudes. We represent the total amplitude by, $T = T(\lambda_e, \lambda_{\bar{e}}; \lambda_t, \lambda_{\bar{t}})$, where the helicity index for each particle is manifest. Then,

$$\begin{aligned} T(\lambda_e, -\lambda_e; \lambda_t, -\lambda_t) &= -\frac{1}{\beta_z^2}\Big(\, \lambda_e \lambda_t + \cos\theta \,\Big) \\ &\qquad\qquad \Big[\, e\, q_t\, \beta_z^2 + (a + b\lambda_e)(c + d\, \lambda_t \beta_t)\,\Big] \\ T(\lambda_e, -\lambda_e; \lambda_t, \lambda_t) &= -\frac{m_t\, \lambda_t}{E\, \beta_z^2}\Big[\, e\, q_t\, \beta_z^2 + (a + b\lambda_e) c\,\Big], \end{aligned} \tag{28}$$

where $\beta_z = \sqrt{1 - m_z^2/4E^2}$, and $\beta_t = \ |\vec{p}|\ /E$. In obtaining the above expression the following relations were useful,

$$\begin{aligned} \vec{\eta}(-\lambda_e, p_e) \cdot \vec{\eta}(\lambda_t, p) &= (\cos\theta + \lambda_e \lambda_t) \\ \hat{p} \cdot \vec{\eta}(-\lambda_e, p_e) &= \sin\theta \\ \hat{p}_e \cdot \vec{\eta}(-\lambda_e, p_e) \times \vec{\eta}(\lambda_t, p) &= -i(\lambda_t + \lambda_e \cos\theta). \end{aligned} \tag{29}$$

The $\gamma\gamma \to t\bar{t}$ Process:

As another example we consider the process $\gamma\gamma \to t\bar{t}$ which will be present in the back-scattered laser mode of an e^+e^- collider. This experiment will have the possibility of polarizing both photons and of analyzing the polarization of the final state quarks.

The relevant diagrams are those obtained via a t-quark exchanged in the t-channel (fig. 2). A straightforward application of (9) or (17) in the CM frame (27) leads to the following expression for the amplitude T,

$$T(h_1, h_2, \lambda_t, \lambda_{\bar{t}}) = \frac{q_t^2}{2}\left[\frac{T^{ij}_{(+)}(h_1, h_2, \lambda_t, \lambda_{\bar{t}})}{p\cdot k_1} + \frac{T^{ij}_{(-)}(h_1, h_2, \lambda_t, \lambda_{\bar{t}})}{p\cdot k_2}\right]\vartheta_1^i \vartheta_2^j \tag{30}$$

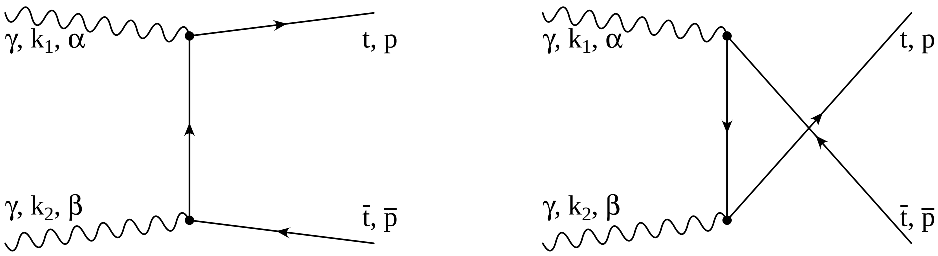

Figure 2: Feynman diagrams for the process $\gamma\gamma \to t\bar{t}$

where $\lambda_t, \lambda_{\bar{t}}$ denote the top and anti-top helicities and $\vartheta_{1,2}$ denote the photon polarization vectors,

$$\vec{\vartheta}_1 = \frac{1}{\sqrt{2}}(1, ih_1, 0); \quad \vec{\vartheta}_2 = \frac{1}{\sqrt{2}}(1, -ih_2, 0), \tag{31}$$

denoting by $h_{1,2} = \pm 1$ the photon helicities. We also defined

$$\begin{aligned} T^{ij}_{(\pm)}(h_1, h_2, \lambda_t, \lambda_t) &= 2m_t \left[\frac{\lambda_t}{|\vec{p}|} \left(2p^i p^j \pm \vec{k}_1 \cdot \vec{p}\, \delta^{ij} \right) + i\epsilon^{ijl}\, k_1^l \right] \\ T^{ij}_{(\pm)}(h_1, h_2, \lambda_t, -\lambda_t) &= 2E \left[p^i \left(\eta^{\lambda_t} \right)^j + p^j \left(\eta^{\lambda_t} \right)^i \pm \vec{k}_1 \cdot \vec{\eta}^{\lambda_t}\, \delta^{ij} \right] \end{aligned} \tag{32}$$

In terms of the momenta in (27) we obtain,

$$\begin{aligned} T(h_1, h_1, \lambda_t, \lambda_{\bar{t}}) &= q_t^2\, \frac{m_t}{E}\, \frac{2}{1 - \beta_t^2 \cos^2\theta} \left(h_1 + \lambda_t \beta_t \right) \delta_{\lambda_t \lambda_{\bar{t}}} \\ T(h_1, -h_1, \lambda_t, \lambda_{\bar{t}}) &= q_t^2\, \frac{2\,\beta_t\,\lambda_t \sin\theta}{1 - \beta_t^2 \cos^2\theta} \begin{cases} \dfrac{m_t}{E} \sin\theta & \text{if } \lambda_{\bar{t}} = \lambda_t, \\ (h_1 + \lambda_t \cos\theta) & \text{if } \lambda_{\bar{t}} = -\lambda_t. \end{cases} \end{aligned} \tag{33}$$

here β_t denotes the velocity of the top-quark, $\beta_t = |\vec{p}| / E$, and θ is the CM scattering angle. Note that as $E \to \infty$, $T(\lambda_t, \lambda_{\bar{t}}, \pm, \mp) \to 0$ as required by helicity conservation. Also in the non-relativistic limit the amplitude for two photons into two like-spin electrons vanishes, as it should.

4 Concluding Remarks

We have presented a new method for computing helicity amplitudes which is applicable for processes involving both massless and massive fermions. The method allows one to obtain a covariant expression for the helicity amplitude without having to go through the process of squaring the amplitude. The method can be easily adopted to both numerical and symbolic computations. We presented two examples which illustrate the simplicity of the method. Although for our two examples we worked in the center of mass frame the method is applicable in any frame and can be used in processes with many pairs of external massive fermions. In particular the methods presented here were used to compute the cross sections for $gg \to t\bar{t}H$, $gg \to t\bar{t} + b\bar{b}$, $gg \to \gamma\gamma t\bar{t}$, and many others (see for example references [14, 15]). For very complicated cases the combination of this method with a program for algebraic manipulation (such as Form, Reduce, Macyma, Maple, or Mathematica) should greatly simplify calculations. Even in those cases where one is only interested in the total cross section the use of the methods presented here may, in some cases, prove simpler to use than the traditional approach.

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6 Appendix

In this appendix we present a proof of the Michel-Bouchiat identity (4). We follow here the notation of reference [16].

The state of a particle can be characterized by giving the momentum $\vec{p}$, mass m, total spin s, and its component of spin along a fixed direction. It is

common to take the z-axis or the direction of the momentum to specify this fixed direction. The latter case is refered to as the helicity basis. In what follows we will use the helicity basis. The advantages of using the helicity basis are well known[1] and we will not dwell on them here.

The helicity operator is defined by,

$$\Lambda \equiv -\frac{2W \cdot \hat{n}}{m}$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma}$ is the Pauli-Lubanski four vector; $J_{\mu\nu}$ and P_μ are the angular momentum and energy-momentum operators respectively. The commutation relation for the W's is,

$$[W^\mu, W^\nu] = -i\epsilon^{\mu\nu\rho\sigma} p_\rho W_\sigma.$$

This relation follows from the commutation properties of $J_{\mu\nu}$ and P_μ. The four vector n is any space-like normalized four vector with $n^2 = -1$. In the helicity basis $\vec{n}$ is in the direction of $\vec{p}$. A conventional form for n is,

$$n_\mu = (\frac{|\vec{k}|}{m}, \frac{k^0}{m}\frac{k_i}{|\vec{k}|}).$$

Following Michel[17] we introduce an orthogonal set of tensors[6], $U^\alpha_\mu = (p_\mu/m; \eta^1_\mu, \eta^2_\mu, \eta^3_\mu)$ such that,

$$\begin{aligned} \eta^i \cdot p &= 0 \\ \eta^i \cdot \eta^j &= -\delta^{ij} \\ g^{\mu\nu} U^\alpha_\mu U^\beta_\nu &= g^{\alpha\beta} \end{aligned}$$

If we define, $S^\alpha \equiv -W^\mu U^\alpha_\mu/m$, then it follows that, $W_\mu = S^i \eta^i_\mu$. For a particle of spin s, with $\eta \in p \cdot \eta = 0$, the operator, $W \cdot \eta$, has $(2s+1)$ eigenvalues: $\lambda = \{-s, -s+1, \ldots, s-1, s\}$. These eigenvalues are used to label states of given momenta and spin.

From the commutation relation for the W's and the properties of the η's it is straightforward to show that the commutation relation for the S's obey an $SU(2)$ algebra,

$$[S_i, S_j] = i\epsilon_{ijk} S_k,$$

[6]The use of these tensors was first suggested to one of us by M.E. Peskin.

Furthermore, if we pick η^3 to have the form,

$$\eta^3_\mu = \left(\frac{\vec{p}}{m}, \frac{k^0}{m}\frac{k_i}{|\vec{k}|}\right),$$

then up to a factor of $\frac{1}{2}$, the operator S_3, is just the helicity operator. In the helicity basis states are labeled by the eigenvalues of S_3. The helicity projection operator is given by,

$$\begin{aligned}\Lambda_\pm = & \quad \tfrac{1}{2}(1 \mp W \cdot \eta_3/m) \\ = & \quad \tfrac{1}{2}\left(1 \pm 2S_3\right),\end{aligned}$$

and as usual the lowering and raising operators of helicity are given by,

$$S_\pm = S_1 \pm iS_2.$$

We label the states for particles of spin $s = \frac{1}{2}$ by $u(p, \lambda)$ where,

$$\begin{aligned}2S_3\, u(p,\lambda) = & \quad \lambda\, u(p,\lambda) \\ \not{p}\, u(p,\lambda) = & \quad m\, u(p,\lambda).\end{aligned} \tag{34}$$

Then,

$$u(p,\pm)\, \bar{u}(p,\pm) = \frac{(1 \pm 2S_3)}{2}(\not{p} + m), \tag{35}$$

where we are using a normalization such that, $\bar{u}(p,\lambda)u(p,\lambda) = 2m$.

In order to generalize the above expression to the case where the spinors have different helicity we simply use the raising and lowering operator for helicity,

$$\begin{aligned}u(p,\mp)\, \bar{u}(p,\pm) &= S_\mp\, u(p,\pm)\, \bar{u}(p,\pm) \\ &= S_\mp\, \frac{(1 \pm 2S_3)}{2}(\not{p} + m) \\ &= (S_1 \mp iS_2)(\not{p} + m),\end{aligned}$$

where we have used, $S_\mp\, S_3 = \pm\frac{1}{2}S_\mp$.

To obtain the Michel-Bouchiat form (4) we recall that, $S_i = -W \cdot \eta_i/m$ and use the definition of W_μ to obtain,

$$S_i = \frac{\gamma_5 \not{\eta}_i \not{p}}{2m},$$

and,

$$
\begin{aligned}
u(p,\pm)\,\bar{u}(p,\pm) &= \frac{1 \pm \gamma_5 \not{n}_3}{2}(\not{p}+m), \\
u(p,\mp)\,\bar{u}(p,\pm) &= \gamma_5 \frac{\not{n}_1 \mp i\not{n}_2}{2}(\not{p}+m).
\end{aligned}
$$

Finally using the Pauli matrices, σ^i, these expressions can be written in the shorthand notation of (4).

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